2.3 Filters

Let I be a nonempty set. The *power set* of I is the set

$$\mathcal{P}(I) = \{A : A \subseteq I\}$$

of all subsets of I. A *filter* on I is a nonempty collection $\mathcal{F} \subseteq \mathcal{P}(I)$ of subsets of I satisfying the following axioms:

- Intersections: if $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$.
- Supersets: if $A \in \mathcal{F}$ and $A \subseteq B \subseteq I$, then $B \in \mathcal{F}$.

Thus to show $B \in \mathcal{F}$, it suffices to show

$$A_1 \cap \cdots \cap A_n \subseteq B,$$

for some n and some $A_1, \ldots, A_n \in \mathcal{F}$.

A filter $\mathcal{F}$ contains the empty set $\emptyset$ iff $\mathcal{F} = \mathcal{P}(I)$. We say that $\mathcal{F}$ is *proper* if $\emptyset \notin \mathcal{F}$. Every filter contains I, and in fact $\{I\}$ is the smallest filter on I. An *ultrafilter* is a proper filter that satisfies

- for any $A \subseteq I$, either $A \in \mathcal{F}$ or $A^c \in \mathcal{F}$, where $A^c = I - A$.

2.4 Examples of Filters

(1) $\mathcal{F}^i = \{A \subseteq I : i \in A\}$ is an ultrafilter, called the *principal ultrafilter generated by* i. If I is finite, then every ultrafilter on I is of the form $\mathcal{F}^i$ for some $i \in I$, and so is principal.

(2) $\mathcal{F}^{co} = \{A \subseteq I : I - A \text{ is finite}\}$ is the *cofinite*, or *Fréchet*, filter on I, and is proper iff I is infinite. $\mathcal{F}^{co}$ is not an ultrafilter.

(3) If $\emptyset \neq \mathcal{H} \subseteq \mathcal{P}(I)$, then the *filter generated by* $\mathcal{H}$, i.e., the smallest filter on I including $\mathcal{H}$, is the collection

$$\mathcal{F}^{\mathcal{H}} = \{A \subseteq I : A \supseteq B_1 \cap \cdots \cap B_n \text{ for some } n \text{ and some } B_i \in \mathcal{H}\}$$

(cf. Exercise 2.7(4)). For $\mathcal{H} = \emptyset$ we put $\mathcal{F}^{\mathcal{H}} = \{I\}$.

If $\mathcal{H}$ has a single member B, then $\mathcal{F}^{\mathcal{H}} = \{A \subseteq I : A \supseteq B\}$, which is called the *principal filter generated by* B. The ultrafilter $\mathcal{F}^i$ of Example (1) is the special case of this when $B = \{i\}$.

(4) If $\{\mathcal{F}_x : x \in X\}$ is a collection of filters on I that is linearly ordered by set inclusion, in the sense that $\mathcal{F}_x \subseteq \mathcal{F}_y$ or $\mathcal{F}_y \subseteq \mathcal{F}_x$ for any $x, y \in X$, then

$$\bigcup_{x \in X} \mathcal{F}_x = \{A : \exists x \in X\,(A \in \mathcal{F}_x)\}$$

is a filter on I.