Hey, Bruce.

I gave that same talk about Calculus with Infinitesimals at the Amatyc conference in Boston last month. It went pretty good, but it was actually more fun with you and the others at our regional meeting.

I read the paper you sent me, Calculus is Algebra by Hatcher (1982). I think I pretty much get it, and he is doing some interesting things. Mostly, he's provided an alternate way of thinking about how the hyperreals are constructed.

The usual way of thinking about the hyperreals ${}^*\mathbb{R}$ is as an *ultrapower* of the reals $\mathbb{R}$. As the ring $\mathbb{R}^{\mathbb{N}}$ is a *direct power* of $\mathbb{R}$, the field ${}^*\mathbb{R}$ is an ultrapower of $\mathbb{R}$. An ultrapower is a quotient of a direct power that arises from the congruence relation defined by an *ultrafilter*. An ultrafilter on $\mathbb{N}$ is a special collection of subsets of $\mathbb{N}$. (See the attached page from Goldblatt.) Two sequences in $\mathbb{R}^{\mathbb{N}}$ are equivalent modulo the ultrafilter $\mathcal{F}$ when the set of indices on which they agree is a member of $\mathcal{F}$.

The usual way of constructing the hyperreals ${}^*\mathbb{R}$ is to begin with the Frechet filter on $\mathbb{N}$ which is the collection of cofinite subsets of $\mathbb{N}$. Then, using the axiom of choice, the existence of a nonprincipal ultrafilter $\mathcal{F}$ that contains the Frechet filter is asserted.

Instead of starting with filters, Hatcher starts with ideals. He begins with the ideal *F*, the set of sequences in the ring $\mathbb{R}^{\mathbb{N}}$ that are zero everywhere except at finitely many indices. Then, using the axiom of choice, he asserts the existence of a maximal ideal *M* which contains *F*. Once he has *M*, he has his quotient ring of equivalence classes ${}^*\mathbb{R} = \mathbb{R}^{\mathbb{N}}/M$ which must be a field since *M* is maximal.

However, at this point, in order to establish the other properties of ${}^*\mathbb{R}$, he has to resort to an ultrafilter after all. So, in Definition 4 and Proposition 2 he derives an ultrafilter *S* from the maximal ideal M. That's kinda cool. But the next two pages contain nothing new as he is just using the ultrafilter as usual to establish the order and extension properties of ${}^*\mathbb{R}$. For the best exposition of the ultrapower construction, I suggest Chapters 2 and 3 of Goldblatt.

After Theorems 3 and 4 on page 367, I like the way he goes on to show how you can think of the halo around *x* being written "algebraically" as the coset $x + I$. This leads him to Theorem 5 where, as expected, he must use the completeness of the reals to prove the result. I like the stuff he goes on to do with the bijection between R_0 and R^2. It might be worth playing with that some more. Could one make graphs for example?

After that, there's really nothing new. For me, the most interesting part of the paper was the footnote on page 363. There he cheerfully admits that his set *S* is an ultrafilter and he talks about a natural bijection between filters on $\mathbb{N}$ and ideals in $\mathbb{R}^{\mathbb{N}}$ where ultrafilters correspond to maximal ideals. The "objective-subjective" distinction that he tries to set up is questionable. But he does introduce the idea of an *elementary extension* which is a nice way to think of the transfer principle. He shows that the function and relation extensions ensured by the logical transfer principle can be thought of as canonical extensions from algebra.

At my talk, when you asked me how ring theory connects to the hyperreals, Theorems 3 and 4 on page 367 are the answer that I gave you. After reading this paper, I see there's another answer too. That's the (now obvious) homomorphism from the ring $\mathbb{R}^{\mathbb{N}}$ to the field ${}^{*}\mathbb{R}$ where the kernel *M* must be (1) the equivalence class of the zero sequence, $M = \left[\{0,0,...\}\right]$; and (2) a maximal ideal in the ring $\mathbb{R}^{\mathbb{N}}$.

By thinking of *M* as the set of all sequences that are equivalent to the zero sequence $\{0,0,...\}$ modulo the ultrafilter *S*, it's apparent that Hatcher's ultrafilter S can also be characterized as the collection of all subsets *s* of $\mathbb{N}$ such that *s* is the set of indices at which some sequence in M is identically zero. In Hatcher-like terms, *S* is the set of all "zero-support sets" for all the sequences in *M*.

So, there are two ring homomorphisms in the offing. The first is the homomorphism from the ring $\mathbb{R}^{\mathbb{N}}$ to the field ${}^{*}\mathbb{R}$. This has Hatcher's maximal ideal *M* as its kernel. The second homomorphism is from the ring $\mathbb{L}$ of finite (limited) hyperreals (Hatcher's R_0) to the field $\mathbb{R}$ of reals numbers. This homomorphism has the set $\mathbb{I}$ of infinitesimals as it's kernel, which must be a maximal ideal in $\mathbb{L}$.

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2.3 Filters

Let I be a nonempty set. The *power set* of I is the set

$$\mathcal{P}(I) = \{A : A \subseteq I\}$$

of all subsets of I. A *filter* on I is a nonempty collection $\mathcal{F} \subseteq \mathcal{P}(I)$ of subsets of I satisfying the following axioms:

- Intersections: if $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$.

- Supersets: if $A \in \mathcal{F}$ and $A \subseteq B \subseteq I$, then $B \in \mathcal{F}$.

Thus to show $B \in \mathcal{F}$, it suffices to show

$$A_1 \cap \cdots \cap A_n \subseteq B,$$

for some n and some $A_1, \ldots, A_n \in \mathcal{F}$.

A filter $\mathcal{F}$ contains the empty set $\emptyset$ iff $\mathcal{F} = \mathcal{P}(I)$. We say that $\mathcal{F}$ is *proper* if $\emptyset \notin \mathcal{F}$. Every filter contains I, and in fact $\{I\}$ is the smallest filter on I. An *ultrafilter* is a proper filter that satisfies

- for any $A \subseteq I$, either $A \in \mathcal{F}$ or $A^c \in \mathcal{F}$, where $A^c = I - A$.

2.4 Examples of Filters

(1) $\mathcal{F}^i = \{A \subseteq I : i \in A\}$ is an ultrafilter, called the *principal ultrafilter generated by* i. If I is finite, then every ultrafilter on I is of the form $\mathcal{F}^i$ for some $i \in I$, and so is principal.

(2) $\mathcal{F}^{co} = \{A \subseteq I : I - A \text{ is finite}\}$ is the *cofinite*, or *Fréchet,* filter on I, and is proper iff I is infinite. $\mathcal{F}^{co}$ is not an ultrafilter.

(3) If $\emptyset \neq \mathcal{H} \subseteq \mathcal{P}(I)$, then the *filter generated by* $\mathcal{H}$, i.e., the smallest filter on I including $\mathcal{H}$, is the collection

$$\mathcal{F}^{\mathcal{H}} = \{A \subseteq I : A \supseteq B_1 \cap \cdots \cap B_n \text{ for some } n \text{ and some } B_i \in \mathcal{H}\}$$

(cf. Exercise 2.7(4)). For $\mathcal{H} = \emptyset$ we put $\mathcal{F}^{\mathcal{H}} = \{I\}$.

If $\mathcal{H}$ has a single member B, then $\mathcal{F}^{\mathcal{H}} = \{A \subseteq I : A \supseteq B\}$, which is called the *principal filter generated by* B. The ultrafilter $\mathcal{F}^i$ of Example (1) is the special case of this when $B = \{i\}$.

(4) If $\{\mathcal{F}_x : x \in X\}$ is a collection of filters on I that is linearly ordered by set inclusion, in the sense that $\mathcal{F}_x \subseteq \mathcal{F}_y$ or $\mathcal{F}_y \subseteq \mathcal{F}_x$ for any $x, y \in X$, then

$$\bigcup_{x \in X} \mathcal{F}_x = \{A : \exists x \in X\, (A \in \mathcal{F}_x)\}$$

is a filter on I.